NEW TRENDS IN FOUNDATIONS OF MATHEMATICS

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Turn in foundations of mathematics under the slogans of

HARD ANALYSIS and PROOF MINING.

New emphasis on finitist methods and results caused by mathematical needs.

New tools as the instances of well-known constructions of proof theory.

Work in Tallinn: E. Tõugu, G. Mints, T. Uustalu, S. Tupailo

Work in USA and Europe: T. Tao, G. Kreisel, U. Kohlenbach.

Essential strengthening of results proved in the mainstream mathematics.

Real (finitist, combinatorial) objects that can be completely encoded in computer: natural numbers, rationals, rational polynomials, rational matrices, etc.

Imaginary or **infinitistic** objects that do not allow such coding. A real number: an infinite sequence of approximations.

Foundational program introduced by D. Hilbert, probably the greatest mathematician of 20-th century, required consistency proof by finitist means. It was extended to reduction of working mathematics to its finitist part.

Most of the fundamental systems can be reduced to their own finitist fragments by so-called *functional interpretations*.

At least some of original motives were philosophical: traditional (classical) approach in mathematics had been considered philosophically defective.

The interest in these developments among mainstream mathematicians seemed to be rather mild.

This impression turned out to be wrong.

http://terrytao.wordpress.com/2007/05/23/

soft-analysis-hard-analysis-and

-the-finite-convergence-principle/

(Now part of a book just published by AMS).

"In the field of analysis, it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on one hand, and "soft", "qualitative", or infinitary" analysis on the other. "Hard analysis" is mostly concerned with finite quantities (e.g. cardinality of finite sets, [...]) and their *quantitative* properties (in particular upper and lower bounds). "Soft" analysis, on the other hand, tends to deal with more infinitary objects (e.g. sequences, measurable sets and functions, [...]) and their *qualitative* properties (convergence, boundedness, integrability, completeness, compactness, etc.).

To put it more symbolically, hard analysis is the mathematics of $\epsilon, N, O()$, and $\leq$; soft analysis is the mathematics of $0, \infty, \epsilon$ and $\to$.

The main problem in transition from soft analysis to hard analysis: proofs by contradiction.

Work done in Tallinn at IoC by Enn Tõugu and his colleagues. I joined this work at a relatively late stage.

Philosophy was of some help with the problem of proofs by contradiction. Intuitionists developed special logical systems avoiding such proofs and allowing to extract computer programs from proofs. Using this idea but different techniques the group in IoC developed programming system PRIZ that exists and develops up to this day. My contribution was to prove that planning (program synthesis) algorithm of PRIZ is identical with what is provided by intuitionistic propositional logic. This extended rather significantly the group of people who understood (at least in part) "what PRIZ is about".

PROOF MINING

Initiated by Kreisel, developed by U. Kohlenbach.

Proof theoretic tools applied to getting essential strengthening of results proved in the mainstream mathematics.

New results are interesting to specialists in a given field (mainly non-linear analysis) and new proofs are stated completely in the framework of the field: no need for practitioners to learn any more logic or constructive mathematics.

The need to understand the underlying logical theory arises only when one tries to understand sources of the constructions used in the new proof and the reason why exactly these constructions are used.

Example. Effective proof of existence of the best approximation in L_1 norm.

Example. Szemeredi's Theorem

Theorem (Wan der Waerden). Every coloring of natural number in a finite number of colors contains arbitrary long monochromatic arithmetical progressions.

Theorem (Szemeredy). Every subset S of natural numbers having positive density contains arbitrary long arithmetical progressions.

Proof. Transform S into a measure space where the shift operator is measure preserving. Prove an ergodic theorem for arbitrary measure preserving operators on such spaces. Specialize to get the result.

To get good bounds, T. Gowers obtained a version dealing with much more finite objects.

Still the multi-dimensional analog had only infinitistic proofs which did not suggest any reasonable bounds. T. Tao and T. Gowers obtained bounds based on finitist version of original construction. In the case of Tao the use of finitization tools was quite conscious.

My recent work. New normalization method for logic based on (non-effective) proof of normalization via Gödel's completeness theorem.

Other recent work in Tallinn.

T. Uustalu. Normalization methods for typed intuitionistic systems.

S. Tupailo. Bounds for higher systems of explicit mathematics.